TASKS AND SUPER-TASKS

By J. F. THOMSON

" TO complete any journey you must complete an infinite
number of journeys. For to arrive from A to B you must
first go to from A to A′, the mid-point of A and B, and thence
to A″, the mid-point of A′ and B, and so on. But it is logically
absurd that someone should have completed all of an infinite
number of journeys, just as it is logically absurd that someone
should have completed all of an infinite number of tasks. There-
fore it is absurd to suppose that anyone has ever completed any
journey."

The argument says that to complete a journey you must do
something that is impossible and hence that you can't complete
a journey.

It may seem that this argument is valid ; and then, since the
conclusion is absurd, we must deny one of the premisses. But
which? Each has a certain plausibility. To some, it is more
plausible that you can't complete an infinite number of journeys
than that you must. These people infer the falsity of the
first premiss from the truth of the second premiss and the falsity
of the conclusion. To others it is more plausible that you must
complete an infinite number of journeys than that you can't.
These people infer the falsity of the second premiss from the
truth of the first premiss and the falsity of the conclusion. The
first party says ' You couldn't, but you don't need to ' ; the
second party says ' You must, but you can '.

This division was neatly illustrated in some recent numbers
of ANALYSIS. Professor Max Black[1] argued that the expression
' an infinite number of acts ' was self-contradictory, and thus
affirmed the second premiss. Unfortunately, he was not entirely
convincing in his rejection of the first premiss. Messrs. Richard
Taylor[2] and J. Watling[3] rejected Professor Black's arguments
for the second premiss, and at least part of their reason for doing
so was that they were impressed by the plausibility of the first
premiss. Unfortunately, they were not entirely convincing in
their rejection of the second.

Luckily we need not take sides in this dispute. For the argu-
ment stated above is not valid. It commits the fallacy of equivo-
cation. There is an element of truth in each of the premisses ;

[1] *Achilles and the Tortoise*, Vol. 11 No. 5.
[2] *Mr. Black on Temporal Paradoxes*, Vol. 12 No. 2.
[3] *The Sum of an Infinite Series*, Vol. 13 No. 2.

what the elements of truth are, it is the purpose of this paper to explain.

Let us begin by considering the second premiss. Is it conceivable that someone should have completed an infinite number of tasks? Do we know what this would be like? Let us say, for brevity, that a man who has completed all of an infinite number of tasks (of some given kind) has completed a super-task (of some associated kind). Then, do we know what a super-task is? Do we have this concept?

It is necessary here to avoid a common confusion. It is not in question whether we understand the sentence : The operation so-and-so can be performed infinitely often. On the contrary, it is quite certain that we do. But to say that some operation can be performed infinitely often is not to say that a super-operation can be performed.

Suppose (A) that every lump of chocolate can be cut in two, and (B) that the result of cutting a lump of chocolate in two is always that you get two lumps of chocolate. It follows that every lump of chocolate is infinitely divisible. Now I suppose that one of the assumptions A and B is false. For either a molecule of chocolate is a lump of chocolate, and then A is false, or it is not, in which case the result of cutting some lump of chocolate in two is not a lump of chocolate, and then B is false. But the conjunction of A and B is certainly consistent, and so it is certainly conceivable that a lump be infinitely divisible. But to say that a lump is infinitely divisible is just to say that it can be cut into any number of parts. Since there is an infinite number of numbers, we could say : there is an infinite number of numbers of parts into which the lump can be divided. And this is not to say that it can be divided into an infinite number of parts. If something is infinitely divisible, and you are to say into how many parts it shall be divided, you have $\aleph_0$ alternatives from which to choose. This is not to say that $\aleph_0$ is one of them.

And if something is infinitely divisible, then the operation of halving it or halving some part of it can be performed infinitely often. This is not to say that the operation *can have been* performed infinitely often.

The confusion that is possible here is really quite gross, but it does have a certain seductiveness. Where each of an infinite number of things can be done, e.g. bisecting, trisecting, etc. ad inf, it is natural and correct to say : *You can perform an infinite number of operations*. (Cf. " you can do it seven different ways ".) But it is also natural, though incorrect, to want to take the italicised sentence as saying that there is some *one* operation you

can perform whose performance is completed when and only when every one of an infinite set of operations has been performed. (A super-operation). This is perhaps natural because, or partly because, of an apparent analogy. If I say "It is possible to swim the Channel" I cannot go on to deny that it is conceivable that someone *should have* swum the Channel. But this analogy is only apparent. To say that it is possible to swim the Channel is to say that there is some one thing that can be done. When we say that you can perform an infinite number of operations we are not saying that there is some one ("infinite") operation you can do, but that the *set* of operations ("finite" operations) which lie within your power is an infinite set. Roughly speaking : to speak of an infinity of possibilities is not to speak of the possibility of infinity.

So far I have just been saying that a certain inference is invalid. Suppose that we are considering a certain set of tasks —ordinary everyday tasks—and that we have assigned numbers to them so that we can speak of Task 1, Task 2, etc. Then : given that for every *n* Task *n* is possible, we cannot straightway infer that some task not mentioned in the premiss is possible, viz. that task whose performance is completed when and only when for every *n* Task *n* has been performed. I have not been saying (so far) that the conclusion of the argument may not be true. But it seems extremely likely, so far as I can see, that the people who have supposed that super-tasks are possible of performance (e.g. Messrs. Taylor and Watling) have supposed so just because they have unthinkingly accepted this argument as valid. People have, I think, confused saying (1) it is conceivable that each of an infinity of tasks be possible (practically possible) of performance, with saying (2) that is conceivable that all of an infinite number of tasks should have been performed. They have supposed that (1) entails (2). And my reason for thinking that people have supposed this is as follows. To suppose that (1) entails (2) is of course to suppose that anyone who denies thinking (2) is committed to denying (1). Now to deny (1) is to be committed to holding, what is quite absurd, (3) that for any given kind of task there is a positive integer *k* such that it is conceivable that *k* tasks of the given kind have been performed, but inconceivable, logically absurd, that *k* + 1 of them should have been performed. But no-one would hold (3) to be true unless he had confused logical possibility with physical possibility. And we do find that those who wish to assert (2) are constantly accusing their opponents of just this confusion. They seem to think that all

they have to do to render (2) plausible is to clear away any confusions that prevent people from accepting (1). (See the cited papers by Messrs. Taylor and Watling, *passim*.)[1]

I must now mention two other reasons which have led people to suppose it obvious that super-tasks are possible of performance. The first is this. It certainly makes sense to speak of someone having performed a number of tasks. But infinite numbers are numbers ; therefore it must make sense to speak of someone having performed an infinite number of tasks. But this perhaps is not so much a reason for holding anything as a reason for not thinking about it. The second is a suggestion of Russell's. Russell suggested[2] that a man's skill in performing operations of some kind might increase so fast that he was able to perform each of an infinite sequence of operations after the first in half the time he had required for its predecessor. Then the time required for all of the infinite sequence of tasks would be only twice that required for the first. On the strength of this Russell said that the performance of all of an infinite sequence of tasks was only medically impossible. This suggestion is both accepted and used by both Taylor and Watling.

Russell has the air of one who explains how something that you might think hard is really quite easy. (" Look, you do it *this* way ".) But our difficulty with the notion of a super-task is not this kind of difficulty. Does Russell really show us what it would be like to have performed a super-task? Does he explain the concept? To me, at least, it seems that he does not even see the difficulty. It is certainly conceivable that there be an infinite sequence of improvements each of which might be effected in a man's skill. For any number n we can imagine that a man is first able to perform just n tasks of some kind in (say) two minutes, and then, after practice, drugs, or meditation is able to perform $n+1$ of them in two minutes. But this is just not to say that we can imagine that someone has effected all the improvements each of which might be effected. If Russell thought it was he was making the mistake already called attention to. And otherwise his suggestion does not help. For the thing said to be possible, and to explain how a super-task is possible, are the things to be explained. If we have no difficulties with " he has effected all of an infinite number of improvements " we are not

[1] See also Mr. Taylor's criticism, ANALYSIS, Vol. 13, No. 1 of J. O. Wisdom's paper, *Achilles on a Physical Race-Course*, ANALYSIS, Vol. 12 No. 3. I am inclined to think that Dr. Wisdom really does deny (1) and that he supposes that he is committed to this course because he wishes to deny (2).
[2] *The Limits of Empiricism*, P.A.S. 1935-36.

likely to be puzzled by " He has performed an infinite number of tasks."

It may be that Russell had in mind only this. If we can conceive a machine doing something—e.g. calling out or writing down numbers—at a certain rate, let us call that rate *conceivable*. Then, there is obviously no upper bound to the sequence of conceivable rates. For any number n we can imagine a machine that calls out or writes down the first n numbers in just $2 - \dfrac{1}{2^{n-1}}$ minutes. But this again is not to say that we can imagine a machine that calls out or writes down all the numerals in just 2 minutes. An infinity of possible machines is not the possibility of an infinity-machine. To suppose otherwise would again be the fallacy referred to.

So far I have only been trying to show that the reasons one might have for supposing super-tasks possible of performance are not very good ones. Now, are there any reasons for supposing that super-tasks are not possible of performance? I think there are.

There are certain reading-lamps that have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and you press the button the lamp goes off. So if the lamp was originally off, and you pressed the button an odd number of times, the lamp is on, and if you pressed the button an even number of times the lamp is off. Suppose now that the lamp is off, and I succeed in pressing the button an infinite number of times, perhaps making one jab in one minute, another jab in the next half-minute, and so on, according to Russell's recipe. After I have completed the whole infinite sequence of jabs, i.e. at the end of the two minutes, is the lamp on or off? It seems impossible to answer this question. It cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction.

This type of argument refutes also the possibility of a machine built according to Russell's prescription that say writes down in two minutes every integer in the decimal expansion of π. For if such a machine is (logically) possible so presumably is one that records the parity, 0 or 1, of the integers written down by the original machine as it produces them. Suppose the parity-machine has a dial on which either 0 or 1 appears. Then, what appears on the dial after the first machine has run through all the integers in the expansion of π?

Now what exactly do these arguments come to? Say that the reading-lamp has either of two light-values, 0 (' off ') and 1 (" on "). To switch the lamp on is then to add 1 to its value and to switch if off is to subtract 1 from its value. Then the question whether the lamp is on or off after the infinite number of switchings have been performed is a question about the value of the lamp after an infinite number of alternating additions and subtractions of 1 to and from its value, i.e. is the question : What is the sum of the infinite divergent sequence
$$+1, \ -1, \ +1, \ \ldots ?$$
Now mathematicians do say that this sequence has a sum ; they say that its sum is $\frac{1}{2}$.[1] And this answer does not help us, since we attach no sense here to saying that the lamp is half-on. I take this to mean that there is no established method for deciding *what* is done when a super-task is done. And this at least shows that the concept of a super-task has not been *explained*. We cannot be expected to *pick up* this idea, just because we have the idea of a task or tasks having been performed and because we are acquainted with transfinite numbers.

As far as I can see the argument given above about the reading-lamp is virtually equivalent to one of Professor Black's arguments.[2] These arguments were however rejected by Taylor and Watling, who said that Black assumed the point at issue by supposing that if any number of tasks have been performed some task of those performed was performed last. This assumption is, they say, exactly the assumption that if any number of tasks have been performed a finite number only have been performed. On the one hand it is not clear to me that Black actually *used* this assumption (clearly he believed it to be true, because it was what he was arguing for) and on the other hand it is clear that the question of a ' last task ' is a little more complicated than Watling and Taylor supposed, just because some infinite sequences really do have last terms as well as first ones. (Thus if you could mention all the positive integers in two minutes in the way that Russell suggests you could also mention them all except 32 in two minutes ; you would then have performed a super-task, but not the super-task of mentioning all the numbers ; but to complete this one you would have only to mention 32, and this would be your last task.) But in any case it should be clear that no assumption about a last task is made in the lamp-argument. If the button has been jabbed an

[1] Hardy, *Divergent Series*, Ch. 1. There is an excellent account of the discussions this series has provoked in Dr. Waismann's *Introduction to Mathematical Thinking*, Ch. 10.
[2] Op. cit. p. 98 para. 17.

infinite number of times in the way described then there was no last jab and we cannot ask whether the last jab was a switching-on or a switching-off. But we did not ask about a last jab ; we asked about the net or total result of the whole infinite sequence of jabs, and this would seem to be a fair question.

It may be instructive here to consider and take quite seriously some remarks of Mr. Watling[1] about summing an infinite convergent sequence. Mr. Watling undertook to show that if you could make all of an infinite number of additions you could compute the sum of the sequence.

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots,$$

that you could add the terms of this sequence together in quite literally the same way as we add together a finite number of numbers, and that you would reach the right answer. For you could add $\tfrac{1}{2}$ to 1, then add $\tfrac{1}{4}$ to the result, and so on, until all of an infinite number of additions have been made. If Mr. Watling were right about this, then of course the net result of at least one super-task would be computable by established methods, indeed by just those methods that we ordinarily use to compute the limit of the sequence of partial sums

$$1, 1\tfrac{1}{2}, 1\tfrac{3}{4}, \ldots,$$

But is he right? There is still the difficulty of supposing that someone could have made an infinite number of additions. (The impossibility of a super-task does not depend at all on whether some vaguely-felt-to-be associated arithmetical sequence is con-vergent or divergent). But besides this, and partly (I think) because of it, there are special difficulties connected with what Mr. Watling says on this score. According to him, " Nothing more is required to give the sum than making every one of the additions." [2] Let us then pretend that we have a machine that does make every one of the additions ; it adds $\tfrac{1}{2}$ to 1 in the first minute of its running time, adds $\tfrac{1}{4}$ to its last result in the next half-minute, and so on. Then this machine does make " every one of the additions ". But does it arrive at the number 2? I do not think we can ask what number the machine arrives at, simply because it does not make a last addition. Every number computed by the machine is a number in the sequence of partial sums given above, i.e. is a value of

$$f(n) = 2 - \frac{1}{2^{n-1}}$$

for some positive integral argument. But to say that the

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[1] Op. cit. pp. 43 et seq.　　　　　　[2] p. 44.

machine computes *only* numbers in this sequence (" Nothing
else is required to give the sum than making every one of the
additions ") *and* that it computes 2 is a flat contradiction. The
result of this pseudo-calculation can only be a term in the
sequence of partial sums ; but equally it cannot be any of these.

This may become clearer if we suppose that the machine
records the results of its successive additions on a tape that runs
through the machine and that the machine only has the voca-
bulary to print terms in the sequence of partial sums. (In
particular the machine cannot print the number 2). The machine
can then record the result of each of the additions that it is
required to make, but it cannot record the number which is
said to be ' finally ' arrived at. Now Mr. Watling would perhaps
wish to say that this machine does in *some* sense ' arrive ' at the
number 2, even though it does not record the fact. But in *what*
sense? What does Mr. Watling mean by the word ' give ' in
the sentence quoted above from his paper? It is surely an
essential feature of our notion of computation (our ordinary
notion of computation) that at some point in the proceedings
the answer to the sum is *read off* ; we find ourselves writing
down or announcing the answer and our algorithm tells us
that this is the answer. It is clear how this is so when we add
together some finite number of numbers. It is not at all clear
from Mr. Watling's paper how or in what sense a man who has
added together an infinite number of numbers can be said to
arrive at his answer. Mr. Watling has not all explained what he
means by saying that the addition of all the terms of the sequence
$1, \frac{1}{2}, \frac{1}{4}, \ldots$ would *yield* 2. And our suspicions on this score should
be increased by his actual ' proof ' that the number reached by
adding together all the terms of an infinite convergent sequence
is the number that is the limit of the sequence of partial sums ;
for this proof is either circular or senseless. Mr. Watling argues
as follows:[1] if you start to add together the terms of the sequence
to be summed, every term you add brings the sum nearer to the
limit of the sequence of partial sums ; therefore, when you have
added together all the terms, there is no difference between the
sum and the limit. This is only to say : the sequence of partial
sums converges to a limit, and therefore the sum of all the terms
in the original sequence *is* the limit. And here Mr. Watling has
given no *independent* sense to " Sum of all the terms ". He refers
us to the steadily-increasing sum of the first so-many terms of the
sequence ; i.e. he refers to the fact that the sequence of partial

[1] Last paragraph on p. 45.

sums is monotonic increasing and convergent. But this gives him no right at all to suppose that he has specified a number to be the sum of all the terms. Mr. Watling supposes, of course, that we can infer what the sum of an infinite sequence is by consideration of the relevant sequence of partial sums; as it is sometimes put, we consider the behaviour of the function that enumerates the partial sums as its argument tends to infinity. And this is quite correct; we can and we do infer what is the sum of an infinite sequence in this way. But this *is* correct *only* because we usually *define* the sum of an infinite sequence to be the limit of the sequence of partial sums. Insofar as Mr. Watling relies tacitly on this his proof is circular. And otherwise I do not think that his proof can be said to show anything at all, for he gives no alternative method whereby the sum of an infinite sequence might be specified.

As far as I can see, we give a sense to the expression " sum of an infinite number of terms " by the methods that we use for computing the limits of certain sequences. There is an inclination to feel that the expression *means* something quite different; that the established method for computing limits is just the way we discover what the sum is, and that the number so discovered can be or should be specified in some other way. But I think that this is just an illusion, born of the belief that one might reach the sum in some other way, e.g. by actually adding together all the terms of the infinite sequence. And I if am correct in supposing that talk of super-tasks is *senseless*, then this kind of talk cannot *give* a sense to anything. The belief that one could add together all the terms of an infinite sequence is itself due presumably to a desire to assimilate sums of infinite sequences to sums of finite ones. This was Mr. Watling's avowed intention. " The limit of a sequence of sums has been called a sum but has not been shown to be one ".[1] Mr. Watling was then trying to *justify* our practice of calling the sum of an infinite sequence a sum. Presumably this practice is due to the fact that the limit of a sequence of sums really is the limit of a sequence of *sums*. If the expression " sum of an infinite sequence " has no meaning apart from the methods we use for computing limits—methods that are, notice, *demonstrably* different from those that we use to compute the sums of finite numbers of terms—this practice *could* not be justified further. The difficulties that Mr. Watling gets into are implicit in his project.[2]

[1] p. 43.
[2] And in fact the project breaks down at once; for Mr. Watling finds it necessary to define the sum of an infinite series ' as the sum of all its terms in a *certain order* and with a *certain grouping*.' (p. 44 my italics). But is not ordinary addition commutative ?

But now, it may be said, surely it is sometimes possible to complete an infinite number of tasks.   For to complete a journey is to complete a task, the task of getting from somewhere to somewhere else. And a man who completes any journey completes an infinite number of journeys. If he travels from 0 to 1, he travels from 0 to $\frac{1}{2}$, from $\frac{1}{2}$ to $\frac{3}{4}$, and so on *ad inf.*, so when he arrives at 1 he has completed an infinite number of tasks. This is virtually the first premiss in the original argument, and it certainly seems both to be true and to contradict the previous result. I think it is true but does not contradict the impossibility of super-tasks.

Let Z be the set of points along the race-course
$$0, \tfrac{1}{2}, \tfrac{3}{4}, \ldots$$
where 0 is the starting-point, 1 the finishing-point ; suppose these on our left and our right respectively. Notice that Z is open on the right ; there is no Z-point to the right of every other Z-point. Z is convergent but its limit-point 1 is not in Z. A point that is neither a Z-point nor to the left of any Z-point I shall call a point external to Z. In particular, 1 is external to Z.

Those who support the first premiss say that all you have to do to get to 1 is to occupy every Z-point from left to right in turn. Or rather they are committed to saying this ; for they do say that to get to 1 it is sufficient to run all the distances in the sequence of distances $\frac{1}{2}$, $\frac{1}{4}$, . . . ; but to occupy every Z-point is to run every one of these distances, since each distance has a right-hand end-point in Z, and, conversely, every Z-point is the right-hand end of one of these distances. But put this way, in terms of points rather than distances, should not their thesis seem odd? For to have arrived at 1 you must have occupied or passed over 1. But 1 is not a Z-point. 1 is not the end-point of any of the distances : first half, third quarter. . . .

Further : suppose someone could have occupied every Z-point without having occupied any point external to Z. Where would he be? Not at any Z-point, for then there would be an unoccupied Z-point to the right. Not, for the same reason, between Z-points. And, ex hypothesi, not at any point external to Z. But these possibilities are exhaustive. The absurdity of having occupied all the Z-points without having occupied any point external to Z is exactly like the absurdity of having pressed the lamp-switch an infinite number of times or of having made all of an infinite number of additions.

But of course those who say that to finish your journey all you have to do is to run each of an infinite number of distances

do not say in so many words that it is possible to have done what was just said to be impossible. And obviously it is possible to have occupied all the Z-points ; you do this by starting off for 1 and making sure you get there. You then occupy a point external to Z; but you have occupied all the Z-points too. Now if you are given a set of tasks, and if it is impossible for you to have performed all the tasks set unless you perform a task not set or not explicitly set, should you not suppose that you are to do something you were not explicitly told to do? (If you are wearing shoes and socks and you are told to take off your socks should you not suppose that you are to take off your shoes?) So if you are told to occupy all the Z-points should you not at once proceed to 1 ?

But the shoes-and-socks analogy is not quite correct. To arrive at 1 you do not have to occupy all the Z-points and then do something else. If you have completed all the journeys that have end-points in Z, there is no further distance to run before arriving at 1. Arriving is not running a last distance. On the contrary, your arriving at 1 is your completing the whole journey and thus is your having completed all the infinite number of journeys (in the only sense in which this is possible). Occupying all the Z-points in turn does not get you to some point short of 1 ; it does not get you, in particular, to a point *next to* 1. Either occupying all the Z-points *is* getting to 1, or it is *nothing*. And this is, perhaps, obscurely noticed by those who support the first premiss. They say, " *all* you need to do is ... " And though it would be permissible to interpret the ' all ' as saying that you need not occupy any point external to Z it could also be interpreted as saying that arriving at 1 is not completing a last journey of those specified.

There is then something odd in the claim that to arrive at 1 you need only occupy all the Z-points. Take it narrowly and it is nonsense. But if we take it charitably, is it not something of a joke? For when the order " Run an infinite number of journeys ! " is so explained as to be intelligible, it is seen to be the order " Run ! " And indeed how could one run any distance without being at some point midway between point of departure and destination? If running to catch a bus is performing a super-task, then this super-task is, for some people at some times, *medically* possible. But *this* super-task is just a task.

If an infinite number of things are to be done, they must be done in some or other order. The order in which they are done imposes an ordering on the set of things to be done. Hence to

the performance of an infinite set of tasks we assign not a trans-finite cardinal but a transfinite ordinal. What is shown by the example of the lamp-switch and by the impossibility of occupying all Z-points without occupying any point external to Z is that it is impossible to have performed every task in a sequence of tasks of type ω (no last task). Now the man who runs from 0 to 1 and so passes over every Z-point may be said to have run every one of an unending sequence of distances, a sequence of type ω. But the proof that he does depends on a statement about arriving at points. Further it is completing a journey that is completing a task, and completing a journey is arriving at a point. And the sequence of points that he arrives at (or is said here to arrive at) is not a sequence of type ω but a sequence of type $\omega + 1$, (last task, no penultimate task) the sequence of the points

$$0, \tfrac{1}{2}, \ldots, 1$$

in Z's closure. So when we explain in what sense a man who completes a journey completes an infinite number of journeys, and thus explain in what sense the first premiss is true, we thereby explain that what is said to be possible by the first premiss is not what is said to be impossible by the second.

The objection to super-tasks was that we could not *say* what would be done if a super-task were done. This objection does not apply in the case of the runner; we can say, he was at 0 and now is at 1. But if it is sometimes possible to have performed all of an infinite sequence of tasks of type $\omega + 1$, why is it not possible to mention all the positive integers except 32 in two minutes, by Russell's prescription, and then mention 32 last? This would be performing a sequence of tasks of type $\omega + 1$. Well, here it would seem reasonable to ask about the state of a parity-machine[1] at the end of the first two minutes, i.e. immedi-ately before the last number was mentioned. But it is obviously unreasonable to ask where the runner was when he was at the point immediately preceding his destination.

There are two points I would like to make in conclusion. There may be a certain reluctance to admitting that one does complete an infinite number of journeys in completing one journey. This reluctance would appear strongly if someone said that the concept of an open point-set like Z was *not applicable to* ' physical reality '. Such a reluctance might be lessened by the following consideration. Let Operation 1 be the operation of proceeding from 0 to the midpoint of 0 and 1. Let Operation 2 be the operation of performing Operation 1 and then pro-ceeding from the point where you are to the midpoint of that

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[1] See above.

point and 1. And in general let Operation n be the operation of performing Operation n–1 and etc. Then Z is the set of points you can get to by performing an operation of the kind described. And obviously none of the operations described gets you to the point 1, hence we should expect Z not to contain its own limit-point and so to be open. Now we just cannot say *a priori* that we shall *never* have occasion to mention point-sets like Z ; one might well want to consider the set of points you can get to by performing operations of this kind. So it is just wrong to say that the concept of an open point-set like Z has no application to ' physical reality ' (which is I think what Black and Wisdom are saying.) But on the other hand the implicit use of the concept in the first premiss of the Zenoesque argument *is* a misleading one, and this is just what the second premiss calls attention to. Roughly speaking the argument forces us to consider the *applications* of the concept of infinity. (E.g. contrast the ways in which it occurs in the propositions I called (1) and (2) at the beginning of this paper.)[1]

Secondly, it may be helpful to indicate the way in which the topic of this discussion is related to the ' mathematical solution ' of the paradox, referred to by all three of the writers I have quoted. People used to raise this topic by asking " *How is it possible* for a man to run all of an infinite number of distances ? " Now either they thought, or Whitehead and others thought they thought, that the difficulty of running an infinite number of distances was like the difficulty of getting to a place an infinite number of miles away. Hence Whitehead emphasised that the sequence

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots$$

was convergent and had a finite sum. He also thereby pointed out a play on the word ' never ' ; the sequence never reaches 0, the sequence of partial sums never reaches 2. (The sequence does not contain its limit ; but it is convergent, the limit exists.)[2] But though this is necessary it does not resolve all the hesitations one might feel about the premiss of the paradox. What I have been trying to show is that these hesitations are not merely frivolous, and that insofar as they spring from misunderstandings these misunderstandings are shared by those who support the ' mathematical solution '.

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[1] I think it is partly this contrast that those people have in mind who claim to distinguish between the concept of a potential infinite and the concept of an actual infinite. But there are not here two concepts but two applications of one concept.

[2] This is clearly explained by Mr. Taylor, op. cit.