ACHILLES AND THE TORTOISE

By MAX BLACK

1. SUPPOSE Achilles runs ten times as fast as the tortoise and gives him a hundred yards start. In order to win the race Achilles must first make up for his initial handicap by running a hundred yards; but when he has done this and has reached the point where the tortoise started, the animal has had time to advance ten yards. While Achilles runs these ten yards, the tortoise gets one yard ahead; when Achilles has run this yard, the tortoise is a tenth of a yard ahead; and so on, without end. Achilles never catches the tortoise, because the tortoise always holds a lead, however small.

This is approximately the form in which the so-called "Achilles" paradox has come down to us. Aristotle, who is our primary source for this and the other paradoxes attributed to Zeno, summarises the argument as follows: "In a race the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead" (Physics, 239^b).[1]

2. It would be a waste of time to prove, by independent argument, that Achilles will pass the tortoise. Everybody knows this already, and the puzzle arises because the conclusion of Zeno's argument is known to be absurd. We must try to find out, if we can, exactly what mistake is committed in this argument.[2]

[1] Aristotle's solution seems to be based upon a distinction between two meanings of 'infinite'—(i) as meaning "infinite in extent", (ii) as meaning "infinitely divisible". "For there are two senses in which length and time and generally anything continuous are called 'infinite': they are called so either in respect of divisibility or in respect of their extremities. So while a thing in a finite time cannot come in contact with things quantitatively infinite, it can come in contact with things infinite in respect of divisibility; for in this sense the time itself is also infinite . . ." (*Physics*, 233^a). This type of answer has been popular (cf. *e.g.* J. S. Mill, *System of Logic*, 5th ed., 389–390). Several writers object that infinite divisibility of the line implies its actually having an infinite number of elements—and so leaves the puzzle unresolved. But see H. R. King, "Aristotle and the paradoxes of Zeno", *Journal of Philosophy* 46 (1949), 657–670.

For references to the vast literature on this and the other arguments of Zeno, see F. Cajori, "The history of Zeno's arguments on motion", *American Mathematical Monthly* 22 (1915), 1–6, 39–47, 77–82, 109–115, 143–149, 179–186, 253–258, 292–297.

[2] It has sometimes been held (*e.g.* by Paul Tannery in *Revue Philosophique* 20 (1885) that Zeno's arguments were sound. "Tannery's explanation of the four arguments, particularly of the 'Arrow' and 'Stade' raises these paradoxes from childish arguments to arguments with conclusions which follow with compelling force . . . it exhibits Zeno as a logician of the first rank" (Cajori, *op. cit.*, 6).

Cf. Russell's remark that the arguments of Zeno "are not, however, on any view, mere foolish quibbles: they are serious arguments, raising difficulties which it has taken two thousand years to answer, and which even now are fatal to the teachings of most philosophers" (*Our Knowledge of the External World* (1926), 175).

3. A plausible answer that has been repeatedly offered[1] takes the line that "this paradox of Zeno is based upon a mathematical fallacy" (A. N. Whitehead, *Process and Reality* (1929), 107).

Consider the lengths that Achilles has to cover, according to our version of the paradox. They are, successively, a hundred yards, ten yards, one yard, a tenth of a yard, and so on. So the total number of yards he must travel in order to catch the tortoise is

$$100 + 10 + 1 + 1/10 + \ldots$$

This is a convergent geometrical series whose sum can be expressed in decimal notion as $111.\dot{1}$, that is to say exactly $111\frac{1}{9}$. When Achilles has run this number of yards, he will be dead level with his competitor; and at any time thereafter he will be actually ahead.

A similar argument applies to the time needed for Achilles to catch the tortoise. If we suppose that Achilles can run a hundred yards in ten seconds, the number of seconds he needs in order to catch up is

$$10 + 1 + 1/10 + 1/100 + \ldots$$

This, too, is a convergent geometrical series, whose sum is expressed in decimal notation $11.\dot{1}$, that is to say exactly $11\frac{1}{9}$. This, as we should expect, is one tenth of the number we previously obtained.

We can check the calculation without using infinite series at all. The relative velocity with which Achilles overtakes the tortoise is nine yards per second. Now the number of seconds needed to cancel the initial gap of a hundred yards at a relative velocity of pursuit of nine yards per second is 100 divided by 9 or $11\frac{1}{9}$. This is exactly the number we previously obtained by summing the geometrical series representing the times needed by Achilles. During this time, moreover, since Achilles is actually travelling at ten yards per second, the actual distance he travels is $10 \times 11\frac{1}{9}$, or $111\frac{1}{9}$, as before. Thus we have confirmed our first calculation by an argument not involving the summation of infinite series.

4. According to this type of solution, the fallacy in Zeno's argument is due to the use of the words "never" and "always". Given the premise that "the pursuer must first reach the point whence the pursued started," it does *not* follow, as alleged, that the quickest runner "never" overtakes the slower: Achilles

[1] In addition to the reference to Whitehead, see for instance Descartes (letter to Clerselier, Adam and Tannery, ed. of *Works* 4, 445–447), and Peirce (Collected Papers, 6.177–6. 182). Peirce says "... this silly little catch presents no difficulty at all to a mind adequately rained in mathematics and in logic ..." (6.177).

does catch the tortoise at some time—that is to say at a time exactly $11\frac{1}{9}$ seconds from the start. It is wrong to say that the tortoise is " always " in front : there is a place—a place exactly $111\frac{1}{9}$ yards from Achilles' starting point—where the two are dead level. Our calculations have showed this, and Zeno failed to see that only a finite time and finite space are needed for the infinite series of steps that Achilles is called upon to make.

5. This kind of mathematical solution has behind it the authority of Descartes and Peirce and Whitehead[1]—to mention no lesser names—yet I cannot see that it goes to the heart of the matter. It tells us, correctly, when and where Achilles and the tortoise will meet, *if* they meet ; but it fails to show that Zeno was wrong in claiming they *could not* meet.

Let us be clear about what is meant by the assertion that the sum of the infinite series.

$$100 + 10 + 1 + 1/10 + 1/100 + \ldots$$

is $111\frac{1}{9}$. It does not mean, as the naive might suppose, that mathematicians have succeeded in adding together an infinite number of terms. As Frege pointed out in a similar connection,[2] this remarkable feat would require an infinite supply of paper, an infinite quantity of ink, and an infinite amount of time. If we had to add all the terms together, we could never prove that the series had a finite sum. To say that the sum of the series is $111\frac{1}{9}$ is to say that if enough terms of the series are taken, the difference between the sum of that *finite number* of terms and the number $111\frac{1}{9}$ becomes, and stays, as small as we please. (Or to put it another way : Let n be any number less than $111\frac{1}{9}$. We can always find a finite number of terms of the series whose sum will be less than $111\frac{1}{9}$ but greater than n).

Since this is all that is meant by saying that the infinite series has a sum, it follows that the " summation " of all the terms of an infinite series is not the same thing as the summation of a finite set of numbers. In one case we can get the answers by working out a finite number of additions ; in the other case we *must* " perform a limit operation ", that is to say, prove that there is a number whose difference from the sum of the initial members of the series can be made to remain as small as we please.

6. Now let us apply this to the race. The series of distances traversed by Achilles is convergent. This means that if Achilles takes enough steps whose sizes are given by the series 100 yards, 10 yards, 1 yard, 1/10 yard, etc. the distance *still to go* to the meeting point eventually becomes, and stays, as small as we

[1] See the last footnote.

[2] *Grundgesetze der Arithmetik* 2 (1903) ; §124. Or see my translation in *Philosophical Review* 59 (1950), 332.

please. After the first step he still has $11\frac{1}{9}$ yards to go ; after the second, only $1\frac{1}{9}$ yard ; after the third, no more than $\frac{1}{9}$ yard ; and so on. The distance still to go is reduced by ten at each move.

But the distance, however much reduced, still remains to be covered ; and after each step there are infinitely many steps still to be taken. The logical difficulty is that Achilles seems called upon to perform *an infinite series of tasks* ; and it does not help to be told that the tasks become easier and easier, or need progressively less and less time in the doing. Achilles may get nearer to the place and time of his rendezvous, but his task remains just as hard, for he still has to perform what seems to be logically impossible. It is just as hard to draw a very small square circle as it is to draw an enormous one : we might say both tasks are infinitely hard. The logical difficulty is not in the extent of the distance Achilles has to cover but in the apparent impossibility of his travelling any distance whatsoever. I think Zeno had enough mathematical knowledge to understand that if Achilles could run $111\frac{1}{9}$ yards—that is to say, keep going for $11\frac{1}{9}$ seconds—he would indeed have caught the tortoise. The difficulty is to understand how Achilles could arrive anywhere at all without first having performed an infinite series of acts.

7. The nature of the difficulty is made plainer by a second argument of Zeno, known as the " Dichotomy " which, according to Aristotle, is " the same in principle " (*Physics*, 239^{b}). In order to get from one point to another, Achilles must first reach a third point midway between the two ; similarly, in order to reach this third point he must first reach a fourth point ; to reach this point he must first reach another point ; and so on, without end. To reach *any* point, he must first reach a nearer one. So, in order to get moving, Achilles must already have performed an infinite series of acts—must, as it were, have travelled along the series of points from the infinitely distant and *open* " end ".[1] This is an even more astounding feat than the one he accomplishes in winning the race against the tortoise.

The two arguments are complementary : the "Achilles " shows that the runner cannot reach any place, even if he gets started ; while the " Dichotomy " shows that he cannot get started, *i.e.* cannot leave any place he has reached.

8. Mathematicians have sometimes said that the difficulty of conceiving the performance of an infinite series of tasks is factitious. All it shows, they say, is the weakness of human

[1] This, at any rate, is the usual interpretation, though I cannot see that Aristotle was thinking of anything more than an argument resembling the "Achilles " in all respects except that of the ratio in which the distance is divided. For the contrary view see, for instance, Sir Thomas Heath, *Mathematics in Aristotle* (1949), 135–6.

imagination and the folly of the attempt to make a mental image of mathematical relationships.[1] The line really does have infinitely many points, and there is no logical impediment to Achilles' making an infinite number of steps in a finite time. I will try to show that this way of thinking about the race is untenable.

9. I am going to argue that the expression, " infinite series of acts ", is self-contradictory, and that failure to see this arises from confusing a series of acts with a series of numbers generated by some mathematical law. (By an " act " I mean something marked off from its surroundings by having a definite beginning and end.)

In order to establish this by means of an illustration I shall try to make plain some of the absurd consequences of talking about " counting an infinite number of marbles ". And in order to do this I shall find it convenient to talk about counting an infinite number of marbles as if I supposed it was sensible to talk in this way. But I want it to be understood all the time that I do not think it sensible to talk in this way, and that my aim in so talking is to show how absurd this way of talking is. Counting may seem a very special kind of " act " to choose, but I hope to be able to show that the same considerations will apply to an infinite series of any kinds of acts.

10. Suppose we want to find out the number of things in a given collection, presumably identified by some description. Unless the things are mathematical or logical entities it will be impossible to deduce the size of the collection from the description alone ; and somebody will have to do the work of taking a census. Of course he can do this without having any idea of how large the collection will turn out to be : his instructions may simply take the form, " Start counting and keep on until there is nothing left in the collection to count ". This implies that there will be a point at which there will be " nothing left to count ", so that the census-taker will then know his task to have been completed.

Now suppose we can know that the collection is infinite. If, knowing this, we were to say " Start counting and continue until there is nothing left to count " we should be practicing a deception. For our census-taker would be led to suppose that sooner or later there would be nothing left to count, while all

[1] " La perception sensible n'embrasse que le fini ; l'imagination atteint encore les infiniment grands et les infiniment petits, tant qu'ils restent finis ; mais elle n'atteint ni l'infini, limite des infiniment grands, ni le *zéro*, limite des infiniment petits : ces deux etats extrémes de la grandeur sont de pures idées, accessibles à la seule raison " (L. Couturat, *De l'infini mathématique* (1896), 562).

the time we would know this supposition to be false. An old recipe for catching guinea pigs is to put salt on their tails. Since they have no tails, this is no recipe at all. To avoid deception we should have said, in the case of the infinite collection, " Start counting and *never* stop ". This should be enough to tell an intelligent census-taker that the collection is infinite, so that there is no sense in trying to count it.

If somebody says to me " Count all the blades of grass in Hyde Park" I might retort "It's too difficult; I haven't enough time ". But if some cosmic bully were to say " Here is an infinite collection; go ahead and count it," only logical confusion could lead me to mutter " Too difficult; not enough time ". The trouble is that, no matter what I do, the result of all my work will not and cannot count as compliance with the instructions. If somebody commands me to obey a certain " instruction ", and is then obliging enough to add that nothing that I can do will count as compliance with that instruction, only confusion could lead me to suppose that any task had been set.

11. Some writers, however, have said that the difficulty of counting an infinite collection is just a matter of *lack of time*.[1] If only we could count faster and faster, the whole job could be done in a finite time ; there would still never be a time at which we were ending, but there would be a time at which we already would have ended the count. It is not necessary to finish counting ; it is sufficient that the counting shall have been finished.

Very well. Since the task is too much for human capacity, let us imagine a machine that can do it. Let us suppose that upon our left a narrow tray stretches into the distance as far as the most powerful telescope can follow; and that this tray or slot is full of marbles. Here, at the middle, where the line of marbles begins, there stands a kind of mechanical scoop ; and to the right, a second, but empty tray, stretching away into the distance beyond the farthest reach of vision. Now the machine is started. During the first minute of its operation, it seizes a marble and transfers it to the empty tray ; then it rests a minute. In the next half-minute the machine seizes a second marble on the left, transfers it, and rests half-a-minute. The third marble is moved in a quarter of a minute, with a corresponding pause ; the next in one eighth of a minute ; and so until the movements are so fast that all we can see is a grey blur. But at the end of exactly

[1] " Quand vous dites qu'une collection infinie ne pourra jamais être numérotée tout entière, il ne s'agit pas là d'une impossibilité intrinsique et logique, mais d'une impossibilité pratique et matérielle : c'est tout simplement une question de temps " (L. Couturat, *op. cit.* (1896), 462).

four minutes the machine comes to a halt, and we now see that the left-hand tray that was full seems to be empty, while the right-hand tray that was empty seems full of marbles.

Let us call this an *infinity machine*. And since it is the first of several to be described let us give it the name "Alpha".

12. I hope nobody will object that the wear and tear on such a machine would be too severe; or that it would be too hard to construct! We are dealing with the logical coherence of ideas, not with the practicability of mechanical devices. If we can conceive of such a machine without contradiction, that will be enough; and believers in the "actual infinite" will have been vindicated.

13. An obvious difficulty in conceiving of an infinity machine is this. How are we supposed to know that there are infinitely many marbles in the left-hand tray at the outset? Or, for that matter, that there are infinitely many on the right when the machine has stopped? Everything we can observe of Alpha's operations (and no matter how much we slow it down!) is consistent with there having been involved only a very large, though still finite, number of marbles.

14. Now there is a simple and instructive way of making certain that the machine shall have infinitely many marbles to count. Let there be only *one* marble in the left-hand tray to begin with, and let some device always return *that same marble* while the machine is resting. Let us give the name 'Beta' to a machine that works in this way. From the standpoint of the machine, as it were, the task has not changed. The difficulty of performance remains exactly the same whether the task, as in Alpha's case, is to transfer an infinite series of qualitatively similar but different marbles; or whether the task, as in Beta's case, is constantly to transfer the *same* marble that is immediately returned to its original position. Imagine Alpha and Beta both at work side by side on their respective tasks: every time the one moves, so does the other; if one succeeds in its task, so must the other; and if it is impossible for either to succeed, it is impossible for *each*.

15. The introduction of our second machine, Beta, shows clearly that the infinite count really is impossible. For the single marble is always returned, and each move of the machine accomplished nothing. A man given the task of filling three holes by means of two pegs can always fill the third hole by transferring one of the pegs; but this automatically creates another empty place, and it won't help in the least to run through this futile series of operations faster and faster. (We

don't get any nearer to the end of the rainbow by running faster). Now our machine, Beta, is in just this predicament: the very act of transferring the marble from left to right immediately causes it to be returned again; the operation is self-defeating and it is logically impossible for its end to be achieved. Now if this is true for Beta, it must be true also for Alpha, as we have already seen.

16. When Hercules tried to cut off the heads of Hydra, two heads immediately grew where one had been removed. It is rumoured that the affair has been incorrectly reported: Zeus, the all powerful, took pity on Hercules and eased his labor. It was decreed that only *one* head should replace the head that had been cut off and that Hercules should have the magical power to slash faster and faster in geometrical progression. If this is what really happened, had Hercules any cause to be grateful? Not a bit. Since the head that was sliced off immediately grew back again, Hercules was getting nowhere, and might just as well have kept his sword in its scabbard.

17. Somebody may still be inclined to say that nevertheless when the machine Beta finally comes to rest (at the end of the four minutes of its operation) the single marble might after all be found in the right-hand tray, and this, if it happened, would *prove* that the machine's task had been accomplished. However, it is easy to show that this suggestion will not work.

I said, before, that "some device" always restored the marble to its original position in the left-hand tray. Now the most natural device to use for this purpose is another machine —Gamma, say—working like Beta but *from right to left*. Let it be arranged that no sooner does Beta move the marble from left to right than Gamma moves it back again. The successive working periods and pauses of Gamma are then equal in length to those of Beta, except that Gamma is working while Beta is resting, and vice versa. The task of Gamma, moreover, is exactly parallel to that of Beta, that is, to transfer the marble an infinite number of times from one side to the other. If the result of the whole four minutes' operation by the first machine is to transfer the marble from left to right, the result of the whole four minutes' operation by the second machine must be to transfer the marble from right to left. But there is only one marble and it must end somewhere! Hence neither machine can accomplish its task, and our description of the infinity machines involves a contradiction.

18. These consideration show, if I am not mistaken, that the outcome of the infinite machine's work is independent of

what the machine is supposed to have done antecedently. The marble might end up on the right, on the left, or nowhere. When Hercules ended his slashing, Zeus had to decide whether the head should still be in position or whether, after all, Hercules' strenuous efforts to do the impossible should be rewarded.

Hercules might have argued that every time a head appeared, he had cut it off, so no head ought to remain; but the Hydra could have retorted, with equal force, that after a head had been removed another had always appeared in its place, so a head ought to remain in position. The two contentions cancel one another and neither would provide a ground for Zeus' decision.

Even Zeus, however, could not abrogate the continuity of space and motion; and this, if I am not mistaken, is the source of the contradiction in our description of the machine Beta. The motion of the marble is represented, graphically, by a curve with an infinite number of oscillations, the rapidity of the oscillations increasing constantly as approach is made to the time at which the machine comes to rest. Now to say that motion is continuous is to deny that any real motion can be represented by a curve of this character. Yet every machine that performed an infinite series of acts in a finite time would have to include a part that oscillated "infinitely fast", as it were, in this impossible fashion. For the beginning of every spatio-temporal act is marked by a change in the velocity or in some other magnitude characterizing the agent.

19. It might be thought that the waiting-intervals in the operations of the three infinity machines so far described have been essential to the argument. And it might be objected that the steps Achilles takes are performed consecutively and without intervening pauses. I will now show that the pauses are not essential.

Consider for this purpose two machines, Delta and Epsilon, say, that begin to work with a single marble each, but in opposite directions. Let Delta start with the marble *a* and Epsilon with the marble *b*. Now suppose the following sequence of operations: while Delta transfers marble *a* from left to right in one minute, Epsilon transfers marble *b* from right to left; then Delta moves *b* from left to right in half a minute while Epsilon returns *a* from right to left during the same time; and so on, indefinitely, with each operation taking half the time of its predecessor. During the time that either machine is transporting a marble, its partner is placing the other marble in position for the next move. Once again, the total tasks of Delta and Epilson are exactly parallel: if the first is to succeed, both marbles must end on the right, but if the second is to succeed, both must end on the left. Hence neither can succeed, and there is a contradiction in our description of the machines.

20. Nor will it help to have a machine—Phi, say—, transferring marbles that become progressively smaller in geometrical progression.[1] For, by an argument already used, we can suppose that while Phi is performing its operations, one of the machines already described is going through its paces at the same rates and at the same times. If Phi could complete its task, Alpha, Beta, Gamma, Delta and Epsilon would have to be able to complete their respective tasks. And we have already seen that this is not possible. The size of the successive tasks has nothing to do with the logical impossibility of performing an infinite series of operations. Indeed it should be clear by this time that the logical possibility of the existence of any one of the machines depends upon the logical possibility of the existence of all of them or, indeed, of any machine that could count an infinite number of objects. If the idea of the existence of any one of them is self-contradictory, the same must be true for each of them. The various descriptions of these different hypothetical devices simply make it easier for us to see that one and all are logically impossible. And though a good deal more needs to be said about this, I hope I have said enough to show why I think this notion of counting an infinite collection is self-contradictory.

21. If we now reconsider for a moment the arguments that have been used in connection with our six infinity machines, we can easily see that no use was made of the respects in which counting differs from any other series of acts. Counting differs from other series of acts by the conventional assignment of numerals to each stage of the count, and it differs in other respects, too. But every series of acts is like counting in requiring the successive doing of things, each having a beginning and end in space or time. And this is all that was used or needed in our arguments. Since our arguments in no way depended upon the specific peculiarities of counting they would apply, as I said at the outset, to any infinite series of acts.

22. And now let us return to Achilles. If it really were necessary for him to perform an infinite number of *acts* or, as Aristotle says " to pass over or severally to come in contact with infinite things " (*Physics*, 233[a]), it would indeed be logically impossible for him to pass the tortoise. But all the things he

[1] An alternative arrangement would be to have three similar machines constantly circulating three marbles.

[1] Somebody might say that if the marble moved by Beta eventually shrunk to nothing there would be no problem about its final location !

really does are finite in number; a finite number of steps, heart beats, deep breaths, cries of defiance, and so on. The track on which he runs has a finite number of pebbles, grains of earth, and blades of grass,[1] each of which in turn has a finite, though enormous number of atoms. For all of these are things that have a beginning and end in space or time. But if anybody says we must imagine that the atoms themselves occupy space and so are divisible " in thought ", he is no longer talking about spatio-temporal things. To divide a thing " in thought " is merely to halve the numerical interval which we have assigned to it. Or else it is to suppose what is in fact physically impossible beyond a certain point, the actual separation of the physical thing into discrete parts. We can of course choose to say that we shall represent a distance by a numerical interval, and that every part of that numerical interval shall also count as respresenting a distance; then it will be true a priori that there are infinitely many " distances ". But the class of what will then be called " distances " will be a series of pairs of numbers, not an infinite series of spatio-temporal things. The infinity of this series is then a feature of one way in which we find it useful to *describe* the physical reality; to suppose that therefore Achilles has to *do* an infinite number of things would be as absurd as to suppose that because I can attach two numbers to an egg I must make some special effort to hold its halves together.

23. To summarise: I have tried to show that the popular mathematical refutation of Zeno's paradoxes will not do, because it simply assumes that Achilles can perform an infinite series of acts. By using the illustration of what would be involved in counting an infinite number of marbles, I have tried to show that the notion of an infinite series of acts is self-contradictory. For any material thing, whether machine or person, that set out to do an infinite number of acts would be committed to performing a motion that was discontinuous and therefore impossible. But Achilles is not called upon to do the logically impossible; the illusion that he must do so is created by our failure to hold separate the finite number of real things that the runner has to accomplish and the infinite series of numbers by which we describe what he actually does. We create the illusion of the infinite tasks by the kind of mathematics that we use to describe space, time, and motion.

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[1] Cf. Peirce: " I do not think that if each pebble were broken into a million pieces the difficulty of getting over the road would necessarily have been increased; and I don't see why it should if one of these millions—or all of them—had been multiplied into an infinity " (*Op. cit.*, 6.182).